LEARNING DYNAMIC SYSTEMS: MARKOV MODELS

Markov Process and Markov Chains

Hidden Markov Models

Kalman Filters

Types of dynamic systems

Problem of future state prediction

Predictability / Observability	Easily predictable state	Hardly predictable state: *state noise*
Easily observable state	Trajectory of a satellite	Trajectory of a GPS localized vehicle
Hardly observable state: *measurement noise*	Controlled underwater vehicle with inertial navigation system in calm water	radiolocalized phones using GSM cells triangulation
Partially observable state + *measurement noise*	Trajectory of an indoor conltroled robot (SLAM)	Phone call localization, speech recognition

Problem of current state estimation (filtering)
Problem of past state estimation (smoothing)
Problem of past state trajectory

Markov models:
a global view

Observability *Type of State*	*Observable state*	*Partially observable state*
Discrete	**Markov Chains**	**Hidden Markov models** (HMM)
Linear continuous models	**Kalman filter,** ARMA models, etc	
Non-linear continuous models	Extended Kalman filter, Particle filters, etc	

LEARNING DYNAMIC SYSTEMS: MARKOV MODELS

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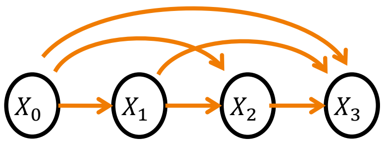
Hidden Markov Models

Bayesian Filtering and Kalman Filter

Markov process

Stochastic process: sequence of random variables $(X_0, \ldots, X_t, \ldots)$

$$P(X_0, \ldots, X_t) = P(X_0) \prod_{i=1}^{t} P(X_i | X_0, \ldots, X_{i-1})$$



Markov process:

« Knowing the past doesn't help to predict the future when the (close) present is known. »

$$P(X_0, \ldots, X_t) = P(X_0) \prod_{i=1}^{t} P\left(X_i \middle| X_{i-1}, \ldots, X_{\max(i-k, 0)}\right)$$

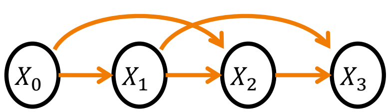
X is the system **state**, k is the **order**.

Examples :

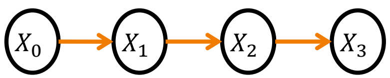
Linear autoregressive models (AR) for prediction in economy/finance

$$X_t = a_1 X_{t-1} + \cdots + a_k X_{t-k} + \varepsilon_t$$

where $\varepsilon_t \sim \mathcal{N}(0, \sigma_t^2)$ is a normal white noise



Markov process of order 2



Markov process (of order 1)

Markov Chain

Definition:

- Markov process of order 1
- with **observable discrete state** (from 1 to n)

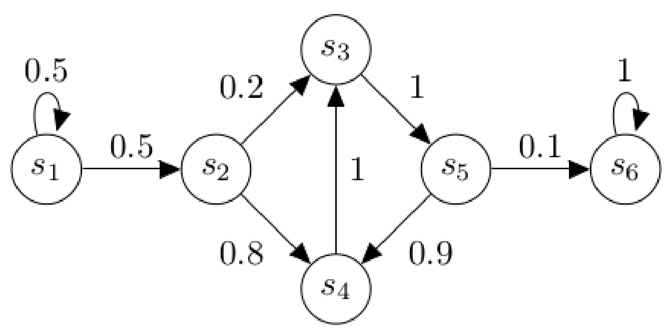
Parameterization: distributions of initial state and **transitions**: $\boldsymbol{\theta} = \left(\left(p_i^0\right)_i, \left(p_{i,j}^t\right)_{\mathrm{t,i,j}} \right)$

$$\boxed{p_i^0 = P(X_0 = i)} \text{ and } \boxed{p_{i,j}^t = P(X_t = i | X_{t-1} = j)}$$

Homogeneous chain: time independent

$$\boxed{p_{i,j} = P(X_t = j | X_{t-1} = i)}$$

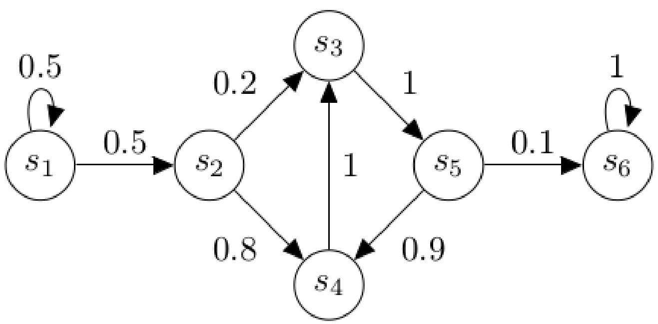
Representation as a graph of a stochastic finite state machine:



Matrix representation of Markov Chains

A Markov chain with n state is defined by a $n \times n$ **transition matrix**:

$$\boxed{\mathbb{P}_t = \left(p_{i,j}^t\right) = \left(P(X_t = j | X_{t-1} = i)\right)_{i,j}}$$



$$\mathbb{P} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Property: $\mathbb{P}_t$ is **stochastic**

$$\forall i, \sum_{j=1}^{n} p_{i,j}^t = 1$$

State prediction from a Markov chain

Fundamental property: given state distribution $P_t = \begin{pmatrix} P(X_t = 1) \\ \vdots \\ P(X_t = n) \end{pmatrix}$,

$$P(X_t = j) = \sum_{i=1}^{n} P(X_t = j | X_{t-1} = i) \cdot P(X_{t-1} = i) \Leftrightarrow \boxed{P_t = \mathbb{P}_t^T \times P_{t-1}}$$

Oubli de la transposition dans le poly!!!

Predicting state at « $t + k$ » :

General case: $P_{t+k} = \prod_{h=1}^{k} \mathbb{P}_{t+h}^T \times P_t$

Homegenous case: $P_{t+k} = {\mathbb{P}^T}^k \times P_t$

Likelihood of an observation $(x_0, \dots x_T)$:

$$P(x_0, \dots x_T | \boldsymbol{\theta}) = P(X_0 = x_0) \prod_{h=1}^{T} (\mathbb{P}_i)_{x_{i-1}, x_i}$$

Learning a Markov Chain

Given n i.i.d state sequences:

$$s_1 = (x_0^1, \dots, x_{T^1}^1), \dots, s_n = (x_0^n, \dots, x_{T^n}^n)$$

What is the underlying Markov Chain?

MLE estimation → frequencies:

$$\hat{p}_{ij}^t = \frac{|k\ s.t.\ x_t^k = j\ and\ x_{t-1}^k = i|}{|k\ s.t.\ x_{t-1}^k = i|}$$

Problem: many coefficients are likely to be equal to $\frac{0}{0}$

→Introduction of a Dirichlet prior: $(p_{ij}^t)_{1\leq j\leq n} \sim Dir\left((\alpha_{ij}^t)_{1\leq j\leq n}\right)$

$$\hat{p}_{ij}^t = \frac{|k\ s.t.\ x_t^k = j\ and\ x_{t-1}^k = i| + \alpha_{ij}^t}{|k\ s.t.\ x_{t-1}^k = i| + \sum_j \alpha_{ij}^t}$$

In general α_{ij}^t=1 (uniform distribution)

Example of application:

One has a corpus of classical music scores with the name of their composer.

1) Problem of supervised classification:

Given a new score, guess the composer.

2) Problem of trajectory generation:

Generate a score that sounds like composed by two given composers

Xenakis and the
« stochastic music »

Solution:

Learn a homogeneous Markov Chain of order k for each composer c

- State space = (pitch, length) of a note
- Compute:

$$\hat{p}^c{}_{i_k,\ldots,i_1,j} = \frac{|t\ s.t. X_t = j\ and\ X_{t-1} = i_1\ \ldots and\ X_{t-k} = i_k| + 1}{|t\ s.t. X_{t-1} = i_1\ \ldots and\ X_{t-k} = i_k| + n}$$

- Find optimal k by cross validation

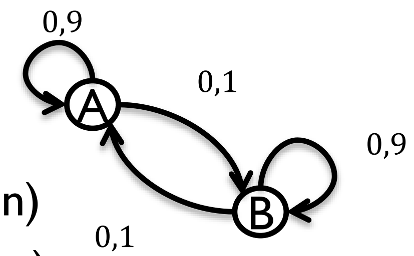
Predict composer for score $(x_0, \ldots x_T)$:

- For each composer, compute likelihood:

$$P(x_0, \ldots x_T | C = c) = p_c(x_0) \prod_{t=1}^{T} \hat{p}^c{}_{x_{t-k,\ldots} x_{t-1}, x_t}$$

- Choose $\hat{c} = \underset{c}{\operatorname{argmax}}\big(P(x_0, \ldots x_T | C = c)\big)$

Problem of trajectory generation:



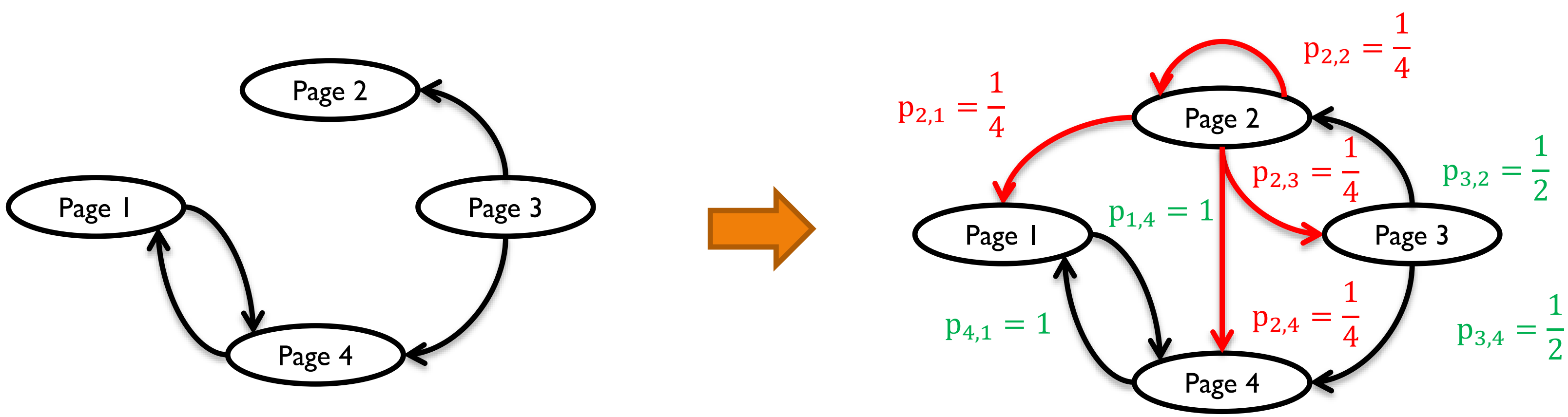
- Averaging both transition matrices is a bad idea (loss of information)
- Use a hierarchical Markov chain with two states (one per composer)
- Generate a note from the Markov chain of the current composer

Other application: PageRank at Google

Compute importance index for nodes in a network (Web, etc)

Let's X_t be the current page of a random Websurfer (random walk model)

Page rank value of page p = asymptotic probability $\lim_{t\to\infty} P(X_t = p)$



Model Web as a Markov Chain $p_{ij} = P(X_t = j | X_{t-1} = i)$

- If $\deg(i) \neq 0$, if "i links to j" $p_{ij} = \frac{1}{\deg(i)}$ otherwise $p_{ij} = 0$
- If $\deg(i) = 0$, $p_{ij} = \frac{1}{n}$

Does $\lim_{t\to\infty} P(X_t = p)$ exist and is independent of X_0?

Stationary distribution and equilibrium

Equilibrium state distribution:
state distribution limit independent of the initial state distribution

$$\exists P_\infty, \forall P_0, \lim_{t \to \infty} P_t = \lim_{t \to \infty} \left({\mathbb{P}^T}^t \times P_0 \right) = P_\infty$$

Stationary distribution : state distribution P_∞ that is a fixed point

$$P_\infty = \mathbb{P}^T \times P_\infty$$

Property 1: an equilibrium distribution is stationary

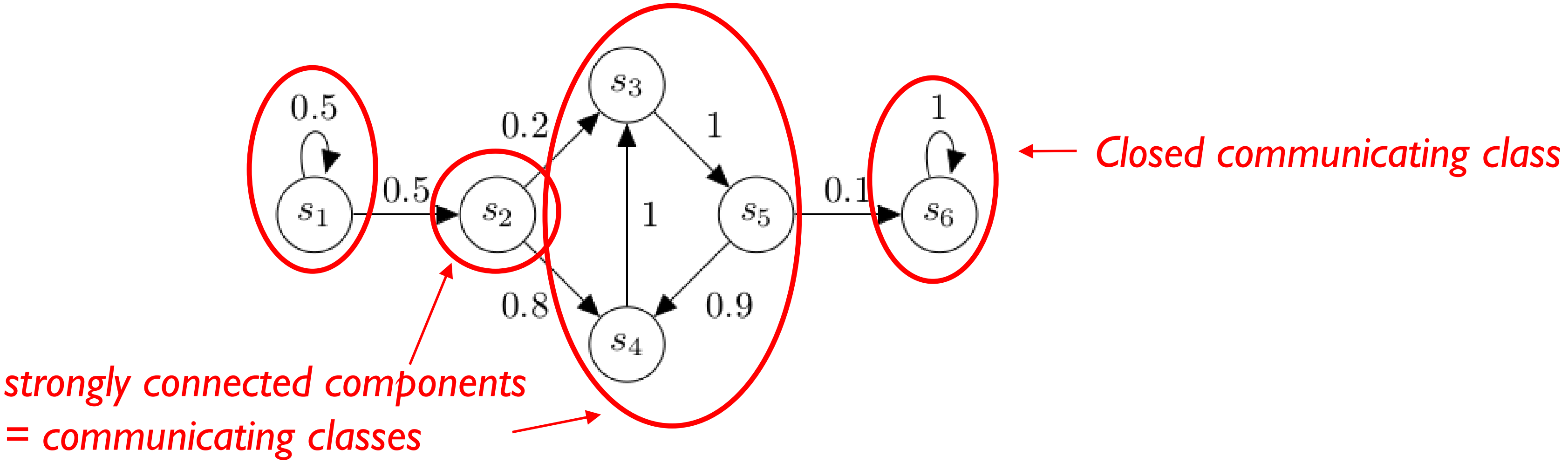
Property 2: every Markov chain has some stationary distribution(s)
Every stochastic matrix accepts 1 as the largest eigenvalue (in absolute value).
Components of right and left eigenvectors for eigenvalue 1 have all the same sign.

$$\mathbb{P}^T = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 1 \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} -0,48 - 0.83i \\ -0,48 + 0.83i \\ 0 \\ 0.5 \\ 0.97 \\ 1 \end{bmatrix} \Rightarrow P_\infty = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Notion of reducibility

- State s' **accessible** from state s ($s \rightarrow s'$) if
$$s \rightarrow s' \Leftrightarrow \exists t \geq 0, P(X_t = s' | X_0 = s) > 0$$
- s and s' **communicate** ($s \leftrightarrow s'$) if $s \rightarrow s'$ and $s' \rightarrow s$
- **Communicating classes** are equivalent classes for $\leftrightarrow$
- A **closed** communicating class has no outgoing link.
- **Theorem:** the number of stationary distributions is the number of closed communicating classes
- A chain is **irreducible** if there is only one (closed) communicating class,

  i.e. the transition graph is **strongly connected**

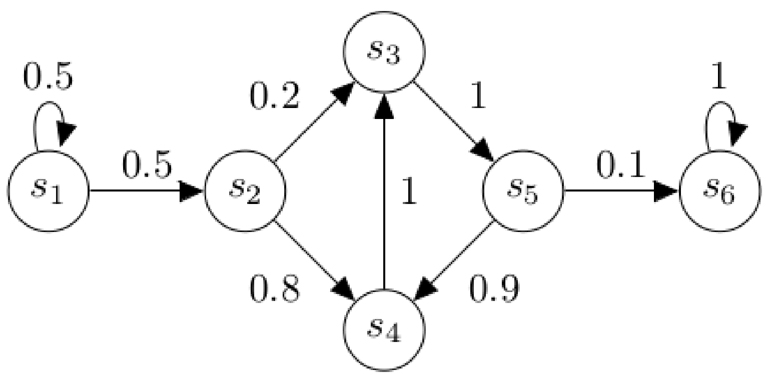


Notion of periodicity

- **Period** of a state is

$$period(s) = \gcd(\{t | P(X_t = s | X_0 = s) > 0\})$$

- A Markov chain is **aperiodic** if all states have a period equal to 1.



$$\Rightarrow periods = \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

Theorem: sufficient condition for convergence to an equilibrium

*An homogeneous **irreducible** and **aperiodic** Markov chain converges to an equilibrium*

Back to PageRank

Problem: the Markov chain of the Web is neither irreducible nor aperiodic

Solution: every **complete** transition graph is irreducible and aperiodic

$$\forall i, \forall j, p_{ij} > 0$$

Algorithm: for every page, one draws a number between 0 and 1

- If $x \geq \alpha$, chooses randomly an outgoing link
- Otherwise, teleport randomly to a page of the Web

Consequence: new transition graph is complete:

$$\forall i, \forall j, p'_{ij} = (1-\alpha)p_{ij} + \alpha \cdot \frac{1}{n} > 0$$

$$\mathbb{P}^T = (1-\alpha)\begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 1 \end{bmatrix} + \frac{\alpha}{n}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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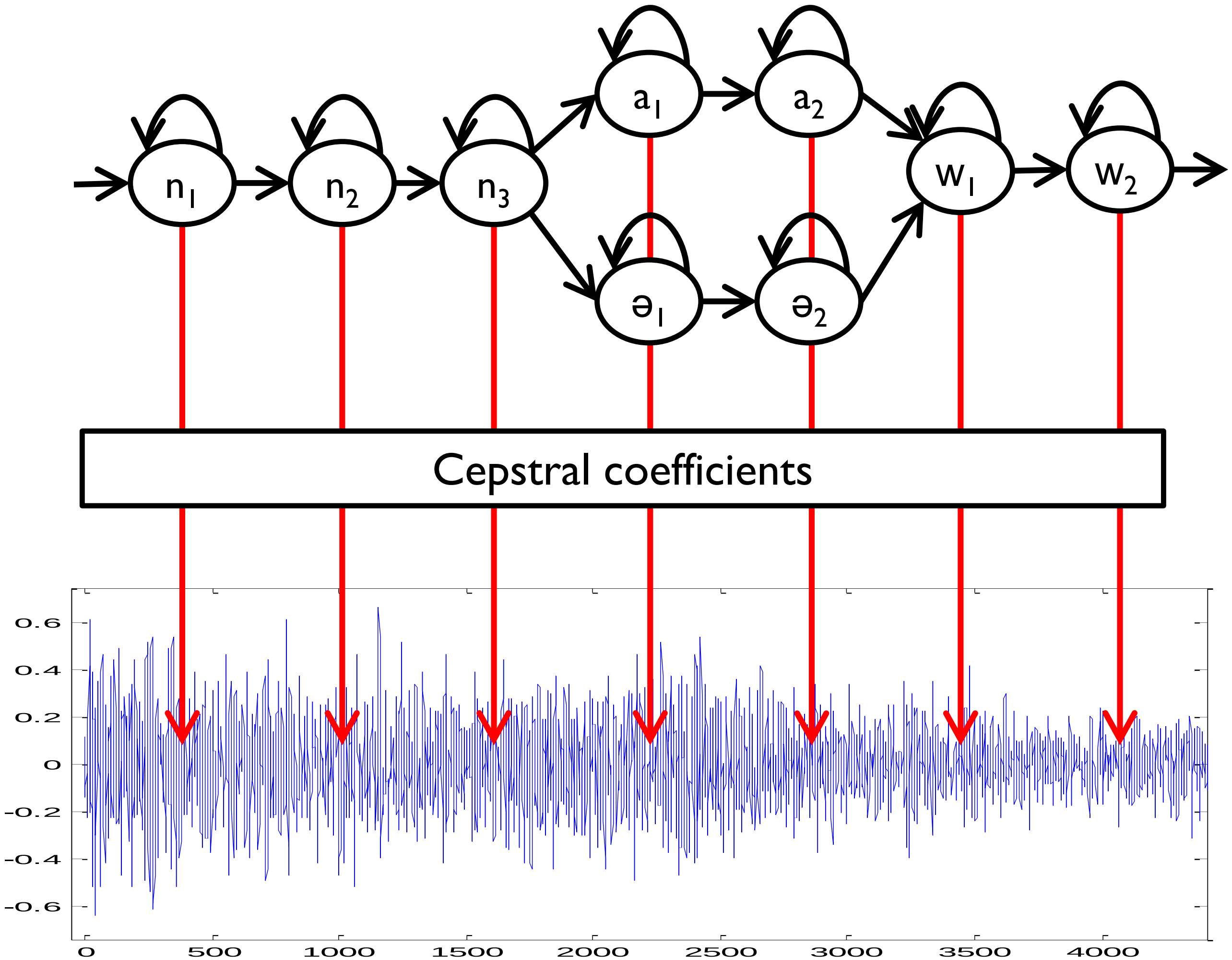
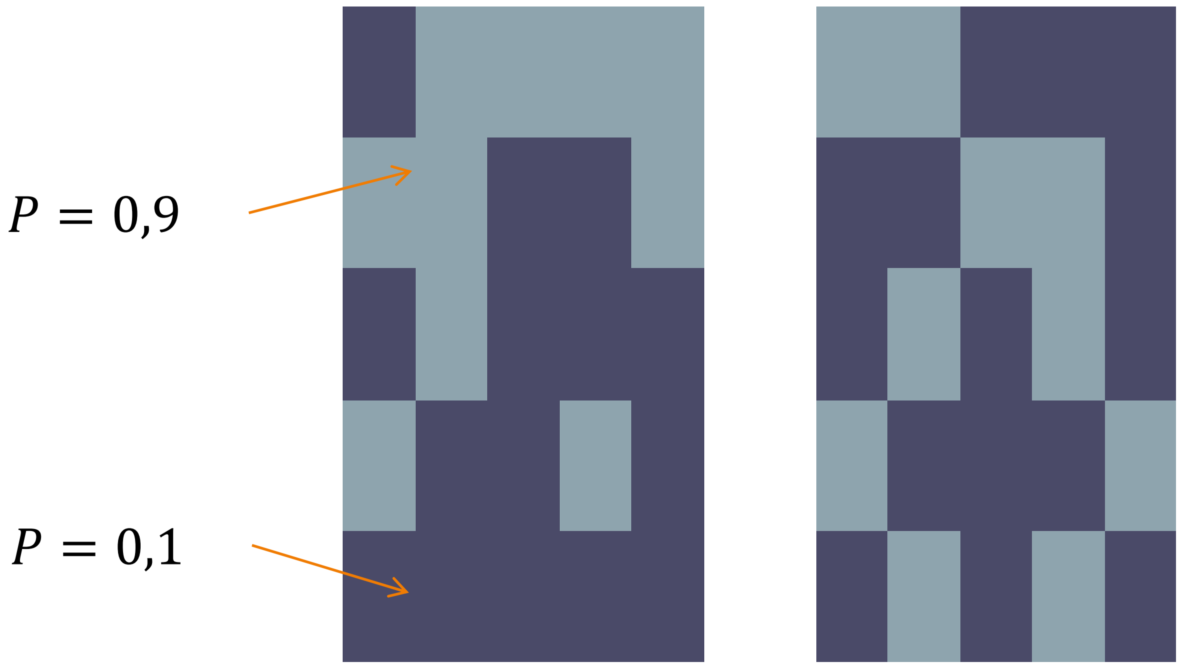
Example of application: Speech recognition systems



from

Example of a partially observable state: the burglar problem (Barber)

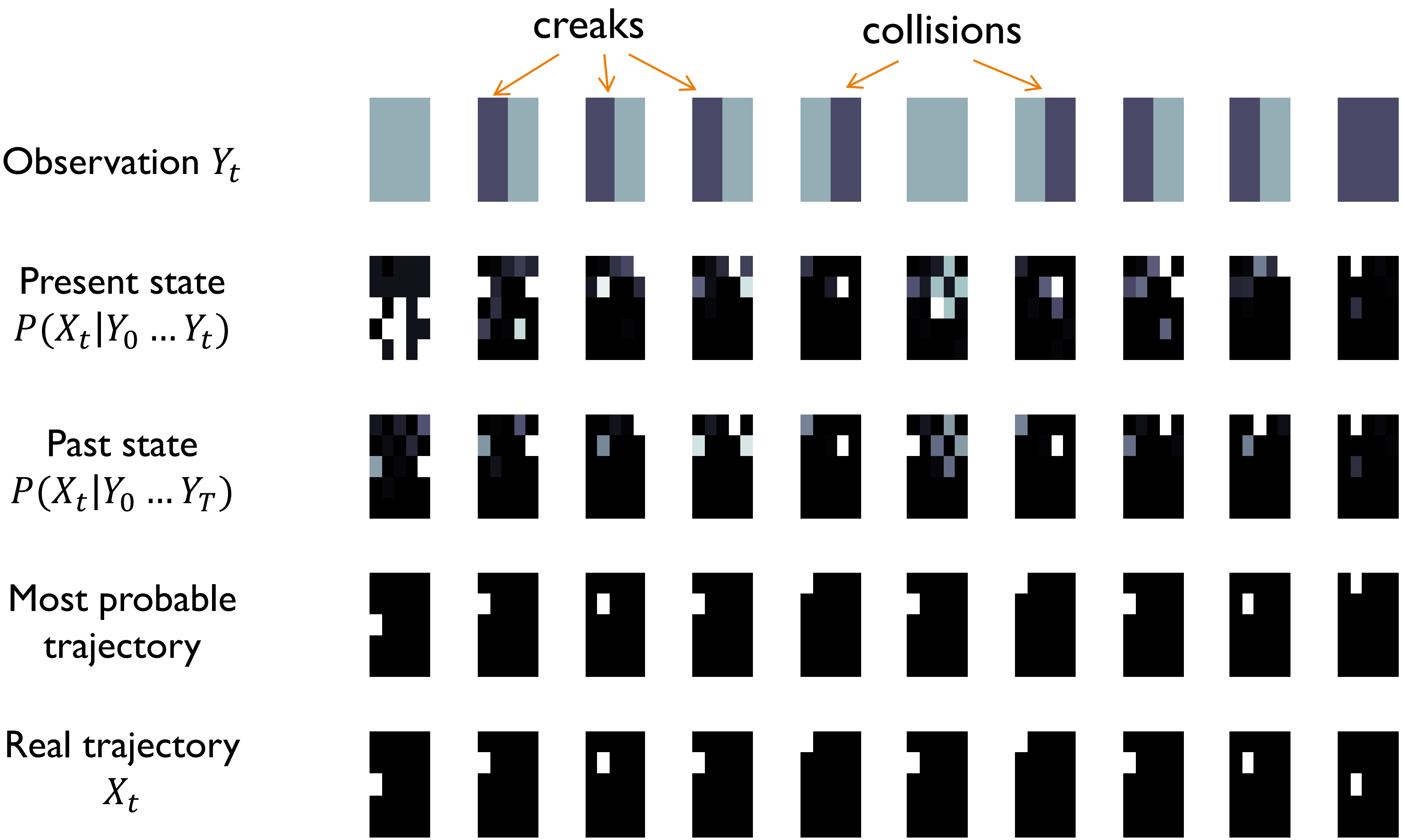
A burglar walks on a grid 5 x 5 in the dark.



Creaking floor

Collision with an obstacle

Different estimation problems with a HMM



Partially observable process

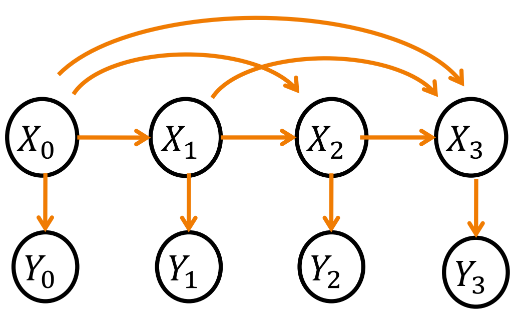
Markov process $\left((X_0, Y_0), \dots, (X_t, Y_t), \dots\right)$ **such that:**

- State X_t is a **hidden variable**.
- State X_t is **partially observable** through **observations** Y_0 à Y_t
- Observation Y_t has only X_t as parent:

$$P(Y_t | X_0, \dots, X_t, \dots) = P(Y_t | X_t)$$

Joint distribution:

$$P(X_0, Y_0, \dots, X_t, Y_t) = P(X_0) P(Y_0 | X_0) \prod_{i=1}^{t} P(X_i | X_0, \dots, X_{i-1}) P(Y_i | X_i)$$



Hidden Markov Model (HMM)

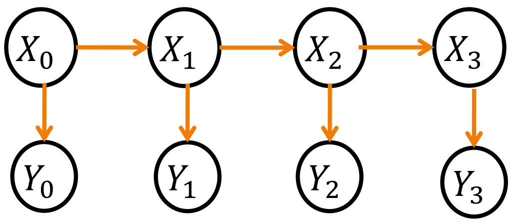
A **Hidden Markov Model** is a partially observable Markov Chain

$$P(X_0, Y_0, \dots, X_t, Y_t) = P(X_0)P(Y_0|X_0)\prod_{i=1}^{t} P(X_i|X_{i-1}, \dots X_{i-k})P(Y_i|X_i)$$

A Hidden Markov Model of order 1 (k=1) with discrete observation from 1 to m is defined by:

- A $n \times n$ **transition matrix**: $\mathbb{P}_t = \left(P(X_t = j|X_{t-1} = i)\right)_{i,j}$
- An $\mathrm{n} \times m$ **emission matrix**: $\mathbb{Q}_t = \left(P(Y_t = j|X_t = i)\right)_{i,j}$

HMM of order 1



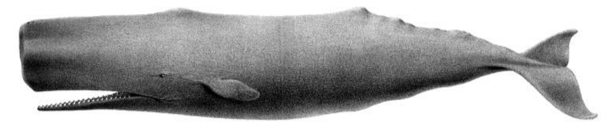
HMM Example: tracking cachalots

Scientifics stick a GPS device on the back of a cachalot:

- A cachalot dives in average 30 min every two hours.
- The device hibernates and wakes up few minutes every 24 hours.
- When woken up, if the device is on the sea surface, it emits its position.
- The risk for the device to become out of service is 5% per day.
- The risk for the device to come off from the cachalot is 10% per day. It then drifts on the surface.
- 50% of the messages are received by a satellite.
- 75% of the messages sent by a drifting device are received by a satellite.
- The average lifetime of the device battery is 100 days with standard deviation of 5 days.

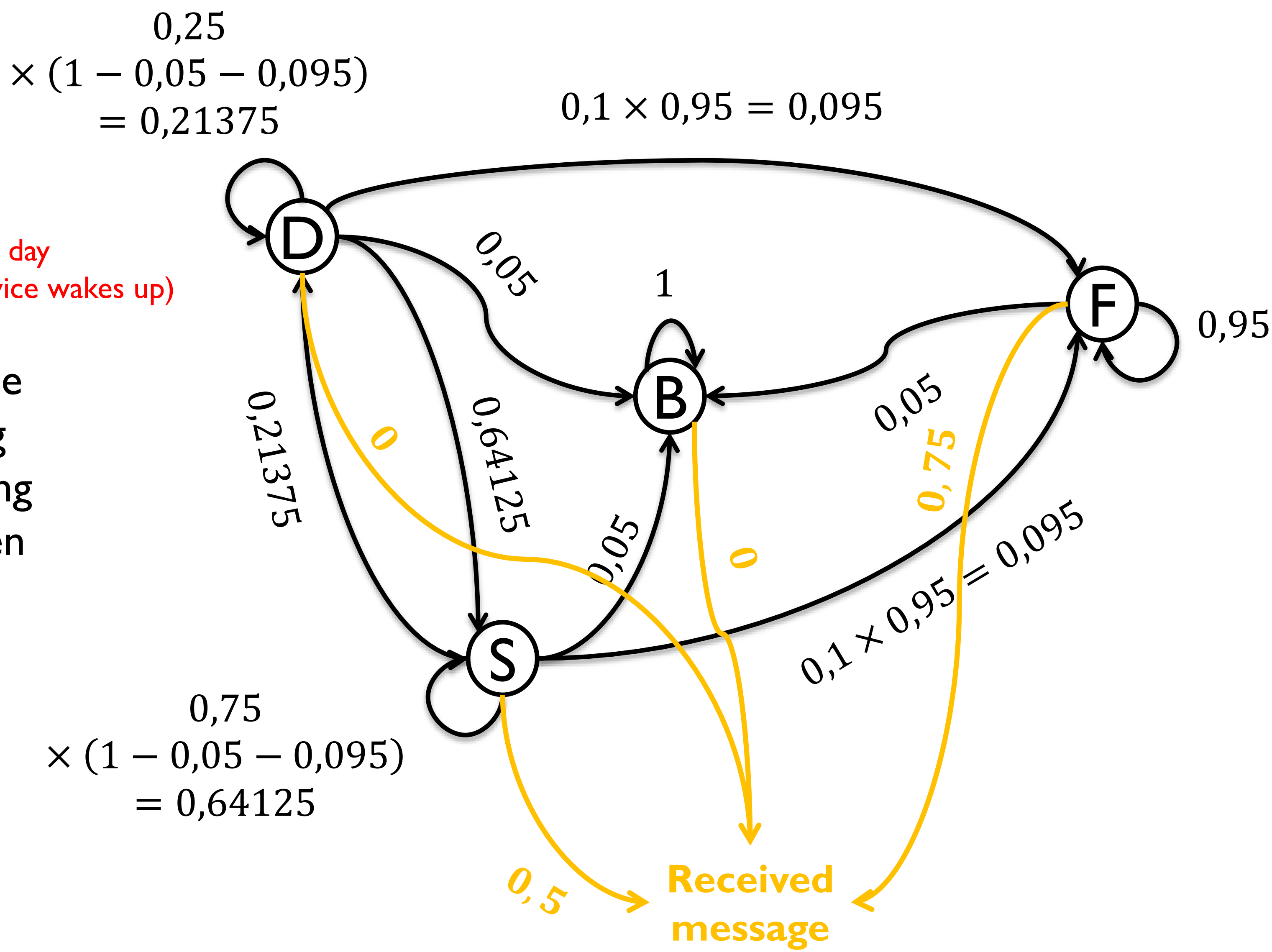
Model the problem with a HMM as a graph and then matrices.

Tracking cachalots: solution

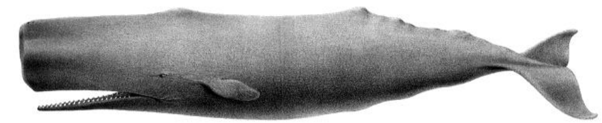


Time step = 1 day
(when the device wakes up)

S Surface
D Diving
F Floating
B Broken



Tracking cachalots: solution



$$\mathbb{P}_t = \begin{array}{c} S \\ D \\ F \\ B \end{array} \begin{bmatrix} 0{,}64125 & 0{,}21375 & 0{,}095 & 0{,}05 \\ 0{,}64125 & 0{,}21375 & 0{,}095 & 0{,}05 \\ 0 & 0 & 0{,}95 & 0{,}05 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P(X_0) = \begin{array}{c} S \\ D \\ F \\ B \end{array} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbb{Q}_t = \begin{array}{c} \\ S \\ D \\ F \\ B \end{array} \begin{bmatrix} M & \neg M \\ 0{,}5 & 0{,}5 \\ 0 & 1 \\ 0{,}75 & 0{,}25 \\ 0 & 1 \end{bmatrix}$$

Online estimation of current state (filtering)

Estimation of the present state $P(X_t|y_0,\dots,y_t)$ from past observations :

$$P(X_t|y_0,\dots,y_t) \quad = \quad \frac{P(X_t,y_0,\dots,y_t)}{P(y_0,\dots,y_t)}$$

$$P(X_t|Y_0,\dots,Y_t) = \frac{\alpha_t(X_t)}{\sum_{X_t}\alpha_t(X_t)} \text{ avec } \alpha_t(X_t) = P(X_t,y_0,\dots,y_t)$$

Recursive « forward » computation of coefficients $\alpha_t(X_t)$:

$$\begin{aligned}
\alpha_t(x) &= P(X_t = x, y_0,\dots,y_t) \\
&= \sum_{x'} P(X_t = x, X_{t-1} = x', y_0,\dots,y_t) \\
&= \sum_{x'} P(y_t|X_t = x)P(X_t = x|X_{t-1} = x')P(X_{t-1} = x', y_0,\dots,y_{t-1}) \\
&= \boxed{P(y_t|X_t = x)\sum_{x'=1}^{n} P(X_t = x|X_{t-1} = x')\alpha_{t-1}(x')}
\end{aligned}$$

$$\boxed{\alpha_t(x) = (\mathbb{Q}_t)_{x,y_t}\cdot(\mathbb{P}_t^T \times \alpha_{t-1})}$$

An example of HMM : on the track of cachalots

Estimate state of device at the 4th day, after positions have been received on days 0, 2 and 3.

$$\alpha_t(x) = (\mathbb{Q}_t)_{x,y_t} \cdot (\mathbb{P}_t^T \times \alpha_{t-1})$$

$$\mathbb{P} = \begin{matrix} S \\ D \\ F \\ B \end{matrix} \begin{bmatrix} 0{,}64125 & 0{,}21375 & 0{,}095 & 0{,}05 \\ 0{,}64125 & 0{,}21375 & 0{,}095 & 0{,}05 \\ 0 & 0 & 0{,}95 & 0{,}05 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$X_0 = \begin{matrix} S \\ D \\ F \\ B \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbb{Q} = \begin{matrix} \\ S \\ D \\ F \\ B \end{matrix} \begin{bmatrix} M & \neg M \\ 0{,}5 & 0{,}5 \\ 0 & 1 \\ 0{,}75 & 0{,}25 \\ 0 & 1 \end{bmatrix}$$

Solution

$\otimes$ *: componentwise multiplication*

$$\alpha_0 = \begin{bmatrix} 0,5 \\ 0 \\ 0,75 \\ 0 \end{bmatrix} \otimes \begin{matrix} S \\ P \\ F \\ D \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0,5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad X_0 | Y_0 = \begin{matrix} S \\ D \\ F \\ B \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\alpha_1 = \begin{bmatrix} 0,5 \\ 1 \\ 0,25 \\ 1 \end{bmatrix} \otimes \mathbb{P}^T \times \begin{bmatrix} 0,5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0,1603 \\ 0,1069 \\ 0,0119 \\ 0,0250 \end{bmatrix} \qquad X_1 | Y_{0\ldots1} = \begin{matrix} S \\ D \\ F \\ B \end{matrix} \begin{bmatrix} 0,53 \\ 0,35 \\ 0,04 \\ 0,08 \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} 0,5 \\ 0 \\ 0,75 \\ 0 \end{bmatrix} \otimes \mathbb{P}^T \times \begin{bmatrix} 0,1603 \\ 0,1069 \\ 0,0119 \\ 0,0250 \end{bmatrix} = \begin{bmatrix} 0,0857 \\ 0 \\ 0,0275 \\ 0 \end{bmatrix} \qquad X_2 | Y_{0\ldots2} = \begin{matrix} S \\ D \\ F \\ B \end{matrix} \begin{bmatrix} 0,76 \\ 0 \\ 0,24 \\ 0 \end{bmatrix}$$

$$\alpha_3 = \begin{bmatrix} 0,5 \\ 0 \\ 0,75 \\ 0 \end{bmatrix} \otimes \mathbb{P}^T \times \begin{bmatrix} 0,0857 \\ 0 \\ 0,0275 \\ 0 \end{bmatrix} = \begin{bmatrix} 0,0275 \\ 0 \\ 0,257 \\ 0 \end{bmatrix} \qquad X_3 | Y_{0\ldots3} = \begin{matrix} S \\ D \\ F \\ B \end{matrix} \begin{bmatrix} 0,52 \\ 0 \\ 0,48 \\ 0 \end{bmatrix}$$

Offline estimation offline of past state (smoothing)

Estimation of $P(X_t|y_0,\dots,y_T)$ given $t \in [\![0 \dots T]\!]$

$$
\begin{aligned}
P(X_t = x|y_0,\dots,y_T) &\propto P(X_t = x, y_0,\dots,y_T) \\
&\propto P(Y_{t+1},\dots,Y_T|X_t = x, y_0,\dots,y_t)P(X_t = x, y_0,\dots,y_t) \\
&\propto P(Y_{t+1},\dots,Y_T|X_t = x)\,\alpha_t(x) \\
&\propto \alpha_t(x)\cdot\beta_t(x)
\end{aligned}
$$

$$
\boxed{P(X_t = x|y_0,\dots,y_T) = \frac{\alpha_t(x)\beta_t(x)}{\sum_x \alpha_t(x)\beta_t(x)}}
$$

with $\boxed{\beta_t(x) = P(y_{t+1},\dots,y_T|X_t = x)}$

Forward/Backward algorithm

In parallel:

1. **Forward recursive computation of** $\forall \boldsymbol{t}, \forall \boldsymbol{x}, \boldsymbol{\alpha_t(x)}$
2. **Backward recursive computation of** $\forall \boldsymbol{t}, \forall \boldsymbol{x}, \boldsymbol{\beta_t(x)}$

$$\begin{aligned}
\beta_t(x) &= P(y_{t+1}, \dots, y_T | X_t = x) \\
&= \sum_{x'} P(y_{t+1}, \dots, y_T, X_{t+1} = x' | X_t = x) \\
&= \sum_{x'} P(y_{t+2}, \dots, y_T | X_{t+1} = x') P(y_{t+1} | X_{t+1} = x') P(X_{t+1} = x' | X_t = x) \\
&= \sum_{x'=1}^{n} P(y_{t+1} | X_{t+1} = x')\, P(X_{t+1} = x' | X_t = x)\, \beta_{t+1}(x')
\end{aligned}$$

$$\boxed{\beta_t(x) = \sum_{x'=1}^{n} (\mathbb{Q}_t)_{x,y_t}\, (\mathbb{P}_{t+1})_{x,x'}\, \beta_{t+1}(x')} \text{ with } \forall x, \beta_T(x) = 1$$

3. **Compute** $\boldsymbol{P(X_t | y_0, \dots, y_T)}$ **from** $\boldsymbol{\alpha_t}$ **and** $\boldsymbol{\beta_t}$

Exemple de HMM : sur les traces des cachalots

Estimate state of device at the 4th day, after positions have been received on days 0, 2 and 3.

$$\boxed{\alpha_t(x) = (\mathbb{Q}_t)_{x,y_t} \cdot (\mathbb{P}_t^T \times \alpha_{t-1})} \text{ and } \boxed{\beta_t(x) = \sum_{x'=1}^{n} (\mathbb{Q}_t)_{x,y_t} \, (\mathbb{P}_{t+1})_{x,x'} \, \beta_{t+1}(x')}$$

$$\mathbb{P} = \begin{matrix} S \\ D \\ F \\ B \end{matrix} \begin{bmatrix} 0{,}64125 & 0{,}21375 & 0{,}095 & 0{,}05 \\ 0{,}64125 & 0{,}21375 & 0{,}095 & 0{,}05 \\ 0 & 0 & 0{,}95 & 0{,}05 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$X_0 = \begin{matrix} S \\ D \\ F \\ B \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbb{Q} = \begin{matrix} \\ S \\ D \\ F \\ B \end{matrix} \begin{bmatrix} M & \neg M \\ 0{,}5 & 0{,}5 \\ 0 & 1 \\ 0{,}75 & 0{,}25 \\ 0 & 1 \end{bmatrix}$$

Solution

$$\alpha_0 = \begin{bmatrix} 0{,}5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \beta_0 = \begin{bmatrix} 0{,}11 \\ 0{,}11 \\ 0{,}12 \\ 0 \end{bmatrix} \qquad X_0|y_{0\ldots3} \propto \alpha_0 \otimes \beta_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad X_0|y_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\alpha_1 = \begin{bmatrix} 0{,}1603 \\ 0{,}1069 \\ 0{,}0119 \\ 0{,}0250 \end{bmatrix} \qquad \beta_1 = \begin{bmatrix} 0{,}18 \\ 0{,}18 \\ 0{,}51 \\ 0 \end{bmatrix} \qquad X_1|y_{0\ldots3} \propto \alpha_1 \otimes \beta_1 = \begin{bmatrix} 0{,}53 \\ 0{,}35 \\ 0{,}12 \\ 0 \end{bmatrix} \qquad X_1|y_{0\ldots1} = \begin{bmatrix} 0{,}53 \\ 0{,}35 \\ 0{,}04 \\ 0{,}08 \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} 0{,}0857 \\ 0 \\ 0{,}0275 \\ 0 \end{bmatrix} \qquad \beta_2 = \begin{bmatrix} 0{,}39 \\ 0{,}39 \\ 0{,}71 \\ 0 \end{bmatrix} \qquad X_2|y_{0\ldots3} \propto \alpha_2 \otimes \beta_2 = \begin{bmatrix} 0{,}63 \\ 0 \\ 0{,}37 \\ 0 \end{bmatrix} \qquad X_2|y_{0\ldots2} = \begin{bmatrix} 0{,}76 \\ 0 \\ 0{,}24 \\ 0 \end{bmatrix}$$

$$\alpha_3 = \begin{bmatrix} 0{,}0275 \\ 0 \\ 0{,}257 \\ 0 \end{bmatrix} \qquad \beta_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad X_3|y_{0\ldots3} \propto \alpha_3 \otimes \beta_3 = \begin{bmatrix} 0{,}52 \\ 0 \\ 0{,}48 \\ 0 \end{bmatrix} \qquad X_3|y_{0\ldots3} = \begin{bmatrix} 0{,}52 \\ 0 \\ 0{,}48 \\ 0 \end{bmatrix}$$

Most probable trajectory: the Viterbi algorithm

Determine most probable sequence of state $(\boldsymbol{x_0}, \dots, \boldsymbol{x_T})$ given observations $(\boldsymbol{y_0}, \dots, \boldsymbol{y_T})$.

$$\operatorname*{argmax}_{x_0 \dots x_T} \mathrm{P}(x_0, \dots, x_T | y_0, \dots, y_T) \propto \operatorname*{argmax}_{x_0 \dots x_T} \mathrm{P}(x_0, \dots, x_T, y_0, \dots, y_T)$$

Resolution by dynamic programming:

$\mu_t(i)$: probability of the most probable trajectory $x_0, \dots, x_t$ such that $x_t = i$.

Bellman equation: $\boldsymbol{\mu_t(i) = \max_{x_0, \dots, x_{t-1}} P(x_0, \dots, x_{t-1}, x_t = i, y_0, \dots, y_t)}$

$$t > 0 \quad \rightarrow \quad \mu_t(i) = P(y_t | X_t = i) \cdot \max_j \left(P(X_t = i | X_{t-1} = j) \mu_{t-1}(j) \right)$$

$$t = 0 \quad \rightarrow \quad \mu_0(i) = P(y_0 | X_0 = i) \cdot P_0(X_0 = i)$$

Final result:

$$\max_{x_0 \dots x_T} \mathrm{P}(x_0, \dots, x_T, y_0, \dots, y_T) = \max_i \left(\mu_T(i) \right)$$

Exemple de HMM : sur les traces des cachalots

Determine most probable trajectory at the 4th day, after positions have been received on days 0, 2 and 3.

$$
\begin{array}{|lll|}
\hline
t > 0 & \rightarrow & \mu_t(i) = P(y_t|X_t = i) \cdot \max_j\big(P(X_t = i|X_{t-1} = j)\ \mu_{t-1}(j)\big) \\
t = 0 & \rightarrow & \mu_0(i) = P(y_0|X_0 = i) \cdot P_0(X_0 = i) \\
\hline
\end{array}
$$

Matrix reformulation where $Diag(v)$ is diagonal matrix of coefs. $v = [c_1, \dots, c_n]$.

$$
\begin{array}{|lll|}
\hline
t > 0 & \rightarrow & \mu_t = Diag(\mathbb{Q}^{*,y_t}) \times \max_j\big(\mathbb{P}^T \times Diag(\mu_{t-1})\big) \\
t = 0 & \rightarrow & \mu_0(i) = Diag(\mathbb{Q}^{*,y_0}) \cdot \mu_0 \\
\hline
\end{array}
$$

Solution

$$\mu_0 = Diag\left(\begin{bmatrix} 0{,}5 \\ 0 \\ 0{,}75 \\ 0 \end{bmatrix}\right) \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{matrix} S \\ P \\ F \\ D \end{matrix} \begin{bmatrix} 0{,}5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mu_1 = Diag\left(\begin{bmatrix} 0{,}5 \\ 1 \\ 0{,}25 \\ 1 \end{bmatrix}\right) \times \max_j \left(\begin{bmatrix} 0{,}64125 & 0{,}64125 & 0 & 0 \\ 0{,}21375 & 0{,}21375 & 0 & 0 \\ 0{,}095 & 0{,}095 & 0{,}95 & 0 \\ 0{,}05 & 0{,}05 & 0{,}05 & 1 \end{bmatrix} \times Diag(\mu_0)\right)$$

$$= Diag\left(\begin{bmatrix} 0{,}5 \\ 1 \\ 0{,}25 \\ 1 \end{bmatrix}\right) \times \max_j \left(\begin{bmatrix} 0{,}3206 & 0 & 0 & 0 \\ 0{,}107 & 0 & 0 & 0 \\ 0{,}0475 & 0 & 0 & 0 \\ 0{,}025 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0{,}160 \\ 0{,}107 \\ 0{,}012 \\ 0{,}025 \end{bmatrix}$$

Solution

$$\mu_2 = Diag\left(\begin{bmatrix}0{,}5\\0\\0{,}75\\0\end{bmatrix}\right)\times \max_j\left(\begin{bmatrix}0{,}64125 & 0{,}64125 & 0 & 0\\0{,}21375 & 0{,}21375 & 0 & 0\\0{,}095 & 0{,}095 & 0{,}95 & 0\\0{,}05 & 0{,}05 & 0{,}05 & 1\end{bmatrix}\times Diag(\mu_1)\right)$$

$$= Diag\left(\begin{bmatrix}0{,}5\\0\\0{,}75\\0\end{bmatrix}\right)\times \max_j\left(\begin{bmatrix}0{,}1028 & 0{,}068 & 0 & 0\\0{,}034 & 0{,}023 & 0 & 0\\0{,}152 & 0{,}010 & 0{,}0113 & 0\\0{,}008 & 0{,}005 & 0{,}0006 & 0{,}025\end{bmatrix}\right) = \begin{bmatrix}0{,}0514\\0\\0{,}0114\\0\end{bmatrix}$$

$$\mu_3 = Diag\left(\begin{bmatrix}0{,}5\\0\\0{,}75\\0\end{bmatrix}\right)\times \max_j\left(\begin{bmatrix}0{,}64125 & 0{,}64125 & 0 & 0\\0{,}21375 & 0{,}21375 & 0 & 0\\0{,}095 & 0{,}095 & 0{,}95 & 0\\0{,}05 & 0{,}05 & 0{,}05 & 1\end{bmatrix}\times Diag(\mu_2)\right)$$

$$= Diag\left(\begin{bmatrix}0{,}5\\0\\0{,}75\\0\end{bmatrix}\right)\times \max_j\left(\begin{bmatrix}0{,}033 & 0 & 0 & 0\\0{,}011 & 0 & 0 & 0\\0{,}005 & 0 & 0{,}011 & 0\\0{,}003 & 0 & 0{,}0006 & 0\end{bmatrix}\right) = \begin{bmatrix}0{,}0165\\0\\0{,}0081\\0\end{bmatrix}$$

Solution

State \ t	μ_0	μ_1	μ_2	μ_3
S	0.5	0.160	0.051	0.0165
D	0	0.107	0	0
F	0	0.012	0.011	0.0085
B	0	0.025	0	0

$\rightarrow$ Most probable trajectory : $(x_0, x_1, x_2, x_3) = (S, S, S, S)$

Learning a HMM:

Problem: learn $\theta = (P_0, (\mathbb{P}_t, \mathbb{Q}_t)_{t\geq 0})$ from N i.i.d sequences

$$\left(y_0^1 \dots y_{T_1}^1\right) \dots \left(y_0^N \dots y_{T_N}^N\right)$$

Two cases:

- States $\left(x_0^1 \dots x_{T_1}^1\right) \dots \left(x_0^N \dots x_{T_N}^N\right)$ are known (expert annotation)

  → Split problem in two parts:

  1. Learn Markov chain $(P_0, (\mathbb{P}_t)_{t\geq 0})$
  2. Learn emission matrices $(\mathbb{Q}_t)_{t\geq 0}$ using MLE

- States $\left(x_0^1 \dots x_{T_1}^1\right) \dots \left(x_0^N \dots x_{T_N}^N\right)$ are unknown

  → EM must be used to learn distribution of hidden states
  → Baum Welch algorithm

The Baum-Welch algorithm

1. Initialize randomly $\theta = (P_0, (\mathbb{P}_t, \mathbb{Q}_t)_{t\geq 0})$ for n fixed

2. **E-step:** estimate $\left(a_t^i\right)_{\mathrm{i,t}}$ and $\left(\boldsymbol{B}_t^i\right)_{i,t}$ from θ using backward-forward algorithm

   - $\forall i, \forall t \in \{0, \dots, T_i\}, \forall s$, distribution $\boldsymbol{a_t^i}$ of X_t^i

$$\boldsymbol{a_t^i}^s \leftarrow \mathrm{P}\left(X_t^i = s \,\middle|\, y_0^i \dots y_{T_i}^i, \theta\right) \propto \alpha_t(s)\beta_t(s)$$

   - $\forall i, \forall t \in \{0, \dots, T_i\}, \forall s, \forall s'$ distribution $\boldsymbol{B}_t^i$ of transition $X_t^i \to X_{t+1}^i$:

$$\boldsymbol{B}_t^{i\ s,s'} \leftarrow \mathrm{P}\left(X_t^i = s, X_{t+1}^i = s' \,\middle|\, y_0^i \dots y_{T_i}^i, \theta\right) \propto \alpha_t(s)\ \mathbb{P}_t^{s,s'}\ \mathbb{Q}_t^{s',y_{t+1}^i}\ \beta_t(s')$$

3. **M-step:** learn $(P_0, (\mathbb{P}_t, \mathbb{Q}_t)_{t\geq 0})$ from $\left(a_t^i\right)_{\mathrm{i,t}}$ and $\left(\boldsymbol{B}_t^i\right)_{i,t}$

   - $\forall t, \forall s, \forall y$

$$P_0^s \propto \sum_i \boldsymbol{a_0^i}^s \qquad and \qquad \mathbb{Q}_t^{s,y} \propto \sum_i \boldsymbol{a_t^i}^s\, \mathbb{I}\left(y_t^i = y\right)$$

   - $\forall t, \forall s, \forall s'$,

$$\mathbb{P}_t^{s,s'} \propto \sum_i \boldsymbol{B}_t^{i\ s,s'}$$

4. Go to step 2 until convergence of θ